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A remark on an Archimedean square of a triangle associated with an arbelos

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Abstract. We consider squares with Archimedean incircle arising from a triangle associated with an arbelos.

Keywords. arbelos, Archimedean square.

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1. Introduction

We consider an arbelos formed by three semicircles α, β and γ with diameters AO, BO and AB, respectively for a point O on the segment AB, where $|AO| = 2a$ and $|BO| = 2b$ (see Figure 1). Circles of radius $r_{\rm A} = ab/(a+b)$ are said to be Archimedean. The radical axis of α and β is called the axis, and the incircle of the arbelos is denoted by δ.

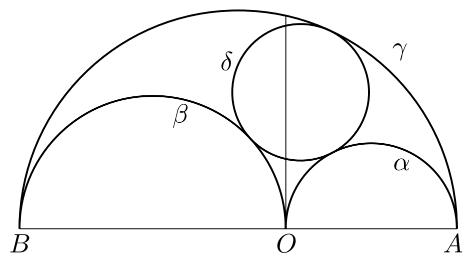

Figure 1.

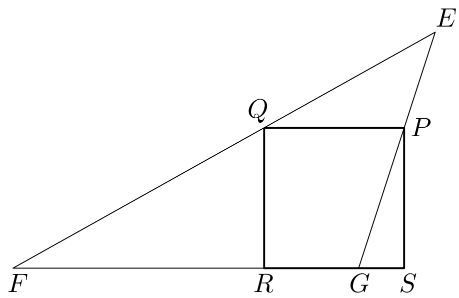

Figure 2.

For a triangle EFG, let $PQRS$ be the square such that the points P and Q lie on the sides GE and EF, respectively, and the side RS lies on the line FG (see Figure 2). We call $PQRS$ the square of EFG on FG. If $|PQ| = 2r_{\rm A}$, the square $PQRS$ is said to be Archimedean. The incircle of an Archimedean square is Archimedean. In this article we construct several triangles whose squares on the base are Archimedean.

2. RESULT

Theorem 1. *For a triangle EFG, let x and y be the length of the base FG and the height, respectively. Then the square of EFG on FG has side length $xy/(x+y)$. Therefore the square is Archimedean if and only if*

$$\frac{xy}{x+y} = 2r_{\mathrm{A}}. \tag{1}$$

Proof. If s is the side length of the square of EFG on FG, we get $(y-s)/s = y/x$ by the similar triangles EFG and EQP. This implies $s = xy/(x+y)$. □

Table 1 shows several pairs of x and y satisfying (1). We now construct triangles with base length x and height y for x and y in the table, where the case 1 is easy since $|AO| = 2a$ and $|BO| = 2b$. We use a rectangular coordinate system with origin O such that the farthest point on α from AB has coordinates (a, a).

Case	x	y
1	$2a$	$2b$
2	$2(a+b)$	$\dfrac{2ab(a+b)}{a^2+ab+b^2}$
3	$a+b$	$\dfrac{2ab(a+b)}{a^2+b^2}$
4	$a,\ (a>b)$	$\dfrac{2ab}{a-b}$
5	$4r_{\mathrm{A}}$	$4r_{\mathrm{A}}$

Table 1. Pairs satisfying (1).

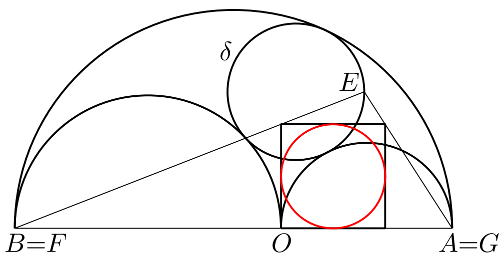

Figure 3.

2.1. **Case 2** $x = 2(a+b)$**,** $y = 2ab(a+b)/(a^2+ab+b^2)$**.** The circle δ has radius $r = ab(a+b)/d$ and center of coordinates $(ab(b-a)/d, 2r)$, where $d = a^2+ab+b^2$. Therefore if E lies on the diameter of δ parallel to AB, and $F = B$ and $G = A$, the square of EFG on FG is Archimedean. The case in which E coincides with the center of δ can be found in [1]. If E coincides with one of the endpoints of the diameter of δ parallel to AB, one of the sides of the square of EFG on FG lies on the axis (see Figure 3).

2.2. **Case 3** $x = a+b$**,** $y = 2ab(a+b)/(a^2+b^2)$**.** If E is the point of tangency of γ and δ, it has coordinates $(2j(b-a), 2j(a+b))$, where $j = ab/(a^2+b^2)$ [2].

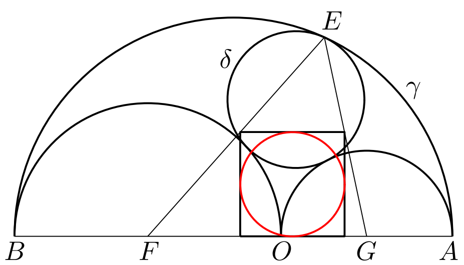

Figure 4.

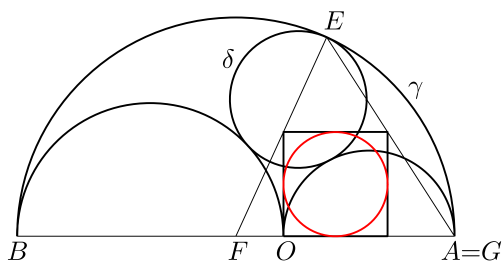

Figure 5.

Therefore if G and F are the centers of α and β, the square of EFG on FG is Archimedean (see Figure 4). Also if F is the center of γ and $G = A$, the square of EFG on FG is Archimedean, and one of the sides of the square lies on the axis (see Figure 5).

2.3. **Case 4** $x = a > b$, $y = 2ab/(a-b)$**.** Assume $a > b$. Let E be the point of intersection of AB and the external common tangent of α and β, which has an equation $(a-b)x - 2\sqrt{ab}y + 2ab = 0$ [3], [4]. If F is the orthogonal projection of the farthest point on α from AB to the axis, and $G = O$, then $|FG| = a$ and $|GE| = 2ab/(a-b)$. Therefore the square of EFG on FG (or GE) is Archimedean (see Figure 6).

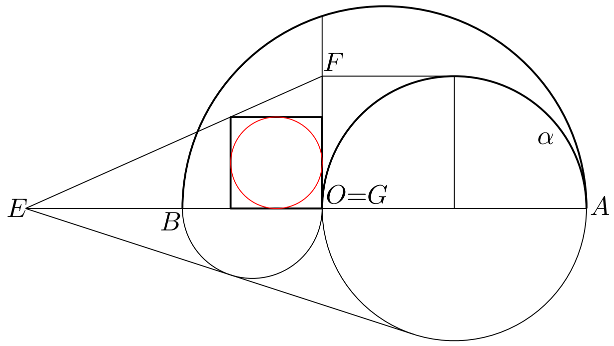

Figure 6.

2.4. **Case 5** $x = 4r_{\mathrm{A}}$, $y = 4r_{\mathrm{A}}$**.** Let ε be the circle touching γ internally and AB at O. Then ε has radius $2r_{\mathrm{A}}$ [4]. Therefore if FG is the orthogonal projection of ε to AB and E is the farthest point on ε from AB, then $|FG| = |EO| = 4r_{\mathrm{A}}$. Hence the square of EFG on FG is Archimedean (see Figure 7). The incircle of the square coincides with Bankoff circle.

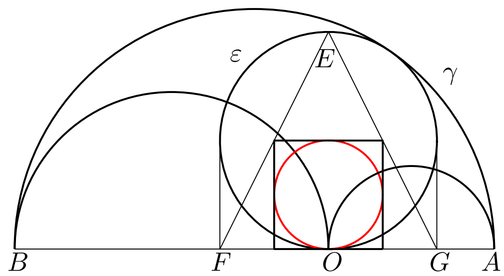

Figure 7.

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